Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz
Association

Signatures and applications in finance

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Memory can determine the dynamics of a stochastic process in different ways, e.g.,

Hidden Markov process: X is a component or function of an underlying Markov process Z. E.g., the price process in a stochastic volatility model

$$dS_t = \sqrt{v_t} S_t dB_t, \quad dv_t = \alpha(v_t)dt + \beta(v_t)dW_t, \quad Z = (S, v).$$

Delay equations: The dynamics of X at time t depends explicitly on $(X_s)_{t-h \le s \le t}$.

Memory kernel: The dynamics of X at time t depends on

$$\int_{-\infty}^{t} K(t,s) X_s ds, \quad \int_{-\infty}^{t} K(t,s) dX_s, \dots$$

Special case: $K(t,s) = K(t-s)$ (Volterra equation).

Processes with memory are the rule, not the exception!

Claim

The path signature is a universal tool for approximating functions of paths, comparable to polynomials in finite dimensions.

1. Introduction to signatures and rough paths (time permitting).
2. Universality of signatures and signature kernels: model-free statistics for stochastic processes.
3. Optimal stopping as an example of using signatures for stochastic optimal control of non-Markov processes.

Paths

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- A path x is smooth if it is C^1 – more precisely, bounded variation would suffice.

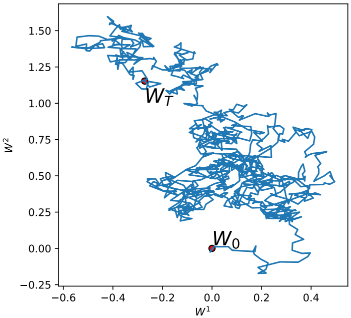

Figure: Sample of a $2d$ Brownian motion W.

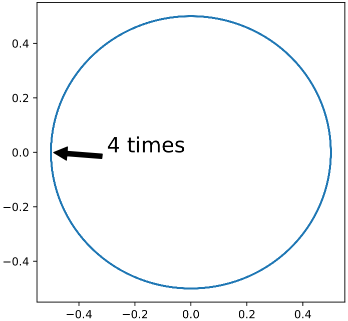

Figure: Path $[0, 1] \ni t \mapsto \frac{1}{4}(\sin(8\pi t), \cos(8\pi t))$.

Controlled differential equation

Let $x : [0, T] \to \mathbb{R}^d$ be a smooth path, $V : \mathbb{R}^e \to \mathbb{R}^{e\times d}$ smooth, $y_0 \in \mathbb{R}^e$, and consider

$$dy(t) = V(y(t))\,dx(t), \quad t \in [0, T], \quad y(0) = y_0.$$

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  $V(y(u)) = V(y(s)) + \text{H.O.T.}$, and hence $y(t) = y(s) + V(y(s))x_{s,t} + \text{H.O.T.}$, $x_{s,t} := x(t) - x(s)$.

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- Second order expansion: $y(u) = y(s) + V(y(s))x_{s,u} +$ H.O.T., implying that

$$V(y(u)) = V(y(s)) + DV(y(s))V(y(s))x_{s,u}, \quad y(t) = y(s) + V(y(s))x_{s,t} + DV(y(s))V(y(s))\mathbb{x}_{s,t} + \text{H.O.T.}$$

$$\mathbb{x}^{(i,j)}_{s,t} := \int_s^t x^i_{s,u}\mathrm{d}x^j(u) = \int_{s<t_1<t_2<t} \mathrm{d}x^i(t_1)\,\mathrm{d}x^j(t_2), \quad i, j = 1, \ldots, d.$$

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- Third order expansion: involves iterated integrals of order three...

Path signature

Given a (smooth) path $x : [s,t] \to \mathbb{R}^d$, the associated signature $\mathbb{x}^{<\infty}_{s,t}$ is the collection of all iterated integrals, i.e., $\mathbb{x}^{<\infty}_{s,t} := \left(\mathbb{x}^{=n}_{s,t}\right)_{n=0}^{\infty}$, where

$$\mathbb{x}^{=0}_{s,t} := 1,\ \mathbb{x}^{=n}_{s,t} := \left(\mathbb{x}^{(i_1,\dots,i_n)}_{s,t}\right)_{(i_1,\dots,i_n)\in\{1,\dots,d\}^n},\ \mathbb{x}^{(i_1,\dots,i_n)}_{s,t} := \int_{s<t_1<\cdots<t_n<t} \mathrm{d}x^{i_1}(t_1)\cdots \mathrm{d}x^{i_n}(t_n).$$

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The signature is parameterization-invariant: i.e., for $\gamma : [u,v] \to [s,t]$ increasing and C^1, the change of variables formula – with $r = \gamma(\bar{r})$ – implies that

$$\int_u^v f(\gamma(\bar{r}))\mathrm{d}x(\gamma(\bar{r})) = \int_u^v f(\gamma(\bar{r}))\dot{x}(\gamma(\bar{r}))\dot{\gamma}(\bar{r})\mathrm{d}\bar{r} = \int_s^t f(r)\dot{x}(r)\mathrm{d}r = \int_s^t f(r)\mathrm{d}x(r).$$

Hence, denoting $z \circ \gamma = x$, we have $\mathbb{z}^{<\infty}_{u,v} = \mathbb{x}^{<\infty}_{s,t}$.

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Open problem

How can we computationally and efficiently recover the path (with unit speed) from its signature?

Tensor algebra

Given a (finite-dimensional) vector space V, let $V^{\otimes 0} := \mathbb{R}$, $V^{\otimes(n+1)} := V^{\otimes n} \otimes V$, and denote

$$T(V) := \bigoplus_{n=0}^{\infty} V^{\otimes n}, \quad T((V)) := \prod_{n=0}^{\infty} V^{\otimes n}, \quad T^N(V) := \bigoplus_{n=0}^{N} V^{\otimes n}$$

Both $T(V)$ and $T((V))$ (and, with obvious modifications, the truncated tensor algebra $T^N(V)$) are algebras with usual addition and the product

$$\mathbf{a} \otimes \mathbf{b} := \left(\sum_{i+j=n} a_i \otimes b_j \right)_{n=0}^{\infty}, \text{ where } \mathbf{a} = (a_n)_{n=0}^{\infty},\ \mathbf{b} = (b_n)_{n=0}^{\infty}.$$

Recall that $\mathbf{a} = (a_n)_{n=0}^{\infty} \in T((V))$ is contained in $T(V)$ iff $a_n = 0 \in V^{\otimes n}$ for all but finitely many n.

- Let $e_1, \ldots, e_d$ denote a basis of $\mathbb{R}^d$, and $x : [s,t] \to \mathbb{R}^d$ a smooth path with $x(u) = \sum_{i=1}^d x^i(u) e_i =: x^i(u) e_i$.
- Recall that $\{ e_{i_1} \otimes \cdots \otimes e_{i_n} \mid (i_1, \ldots, i_n) \in \{1, \ldots, d\}^n \}$ is a basis of $(\mathbb{R}^d)^{\otimes n}$.
- We denote the basis of $(\mathbb{R}^d)^{\otimes 0} \simeq \mathbb{R}$ by $\mathbf{1}$ – which we identify with $(\mathbf{1}, 0, \ldots) \in T((\mathbb{R}^d))$. Note that $\mathbf{1}$ is the neutral element of the algebra $T((\mathbb{R}^d))$ w.r.t. $\otimes$.

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Definition (Path signature)

We define the signature $\mathbb{x}_{s,t}^{<\infty} \in T((\mathbb{R}^d))$ by setting

$$\mathbb{x}_{s,t}^{<\infty} := \mathbf{1} + \sum_{n=1}^{\infty} \sum_{(i_1,\dots,i_n)\in\{1,\dots,d\}^n} \mathbb{x}_{s,t}^{(i_1,\dots,i_n)} e_{i_1} \otimes \cdots \otimes e_{i_n} =: \mathbf{1} + \sum_{n=1}^{\infty} \int_{s<t_1<\cdots<t_n<t} \mathrm{d}x(t_1) \otimes \cdots \otimes \mathrm{d}x(t_n),$$

as well as its truncated version $\mathbb{x}_{s,t}^{\leq N} \in T^N(\mathbb{R}^d)$ by truncation at level N.

Theorem (Chen's identity)

Given a (smooth) path $x : [r,t] \to \mathbb{R}^d$, *then for any* $r < s < t$ *we have*

$$\mathbb{X}^{<\infty}_{r,t} = \mathbb{X}^{<\infty}_{r,s} \otimes \mathbb{X}^{<\infty}_{s,t}.$$

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- Formally, Chen's identity follows easily from the differential equation satisfied by the signature:

$$\mathrm{d}\mathbb{x}^{<\infty}_{s,t} = \mathbb{x}^{<\infty}_{s,t} \otimes \mathrm{d}x(t), \quad \mathbb{x}^{<\infty}_{s,s} = \mathbf{1} \in T((\mathbb{R}^d)).$$

- Chen's identity is a consequence of linearity of the integral. Hence, it is a fundamental property valid for all notions of signatures, including for rough paths.

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- Given two paths $x : [a,b] \to \mathbb{R}^d$ and $y : [c,e] \to \mathbb{R}^d$, define their concatenation product $z := x \circ y : [a, b + (e-c)] \to \mathbb{R}^d$ by

$$z(u) := \begin{cases} x(u), & a \le u \le b, \\ y(u - b + c) - y(c) + x(b), & b < u \le b + (e-c). \end{cases}$$

  By Chen's identity (and re-parameterization invariance), $\mathbb{z}^{<\infty}_{a,b+(e-c)} = \mathbb{x}^{<\infty}_{a,b} \otimes \mathbb{y}^{<\infty}_{c,e}$.
- Let $\overleftarrow{x}$ the time-reversal of x, so $z := x \circ \overleftarrow{x}$ is tree-like, $\mathbb{z}^{<\infty}_{r,t} = \mathbf{1}$. Hence, $\overleftarrow{\mathbb{x}}^{<\infty}_{s,t} = (\mathbb{x}^{<\infty}_{r,s})^{-1}$.

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- Let $\mathcal{W}_d$ denote the linear span of all such words: For words $\mathtt{w}_1,\dots,\mathtt{w}_k$, a typical element $\ell \in \mathcal{W}_d$ is of the form $\ell = \lambda_1 \mathtt{w}_1 + \cdots + \lambda_k \mathtt{w}_k, \quad \lambda_1,\dots,\lambda_k \in \mathbb{R}$.
- Extending the concatenation product in a distributive way to $\mathcal{W}_d$, we obtain an algebra, including the empty word $\varnothing$ as neutral element w.r.t. multiplication (i.e., concatenation).

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Definition (Duality pairing)

Define a bi-linear map $\langle \cdot, \cdot \rangle : \mathcal{W}_d \times T((\mathbb{R}^d)) \to \mathbb{R}$: For a word $\ell = \mathtt{i}_1 \cdots \mathtt{i}_k \in \mathcal{W}_d$, and for

$$T((\mathbb{R}^d)) \ni \mathbf{a} = a^{\varnothing}\mathbf{1} + \sum_{n=1}^{\infty} \sum_{(i_1,\dots,i_n)\in\{\,1,\dots,d\,\}^n} a^{(i_1,\dots,i_n)} e_{i_1} \otimes \cdots \otimes e_{i_n},$$

set $\langle \mathtt{i}_1 \cdots \mathtt{i}_k, \mathbf{a} \rangle := a^{(i_1,\dots,i_k)}$, and extend bi-linearly to $\mathcal{W}_d$ in the first argument.

Definition (Shuffle product)

Define a commutative product $\sqcup\!\sqcup$ on $\mathcal{W}_d$ as follows: For words $\mathtt{w}$, $\mathtt{v}$ and letters $\mathtt{i}$, $\mathtt{j}$ define

$$\mathtt{w} \sqcup\!\sqcup \varnothing := \varnothing \sqcup\!\sqcup \mathtt{w} := \mathtt{w}, \quad \mathtt{wi} \sqcup\!\sqcup \mathtt{vj} := (\mathtt{w} \sqcup\!\sqcup \mathtt{vj})\mathtt{i} + (\mathtt{wi} \sqcup\!\sqcup \mathtt{v})\mathtt{j},$$

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Theorem (Shuffle identity)

Given a smooth path $x : [s,t] \to \mathbb{R}^d$ and $\ell_1, \ell_2 \in \mathcal{W}_d$, we have

$$\left\langle \ell_1, \mathbb{x}^{<\infty}_{s,t} \right\rangle \left\langle \ell_2, \mathbb{x}^{<\infty}_{s,t} \right\rangle = \left\langle \ell_1 \sqcup\!\sqcup \ell_2, \mathbb{x}^{<\infty}_{s,t} \right\rangle.$$

Remarks on the shuffle identity

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$$\left\langle \ell_1 \sqcup\!\sqcup\, \ell_2,\, \mathbb{x}^{<\infty}_{s,t} \right\rangle = 2\left\langle \mathtt{ii},\, \mathbb{x}^{<\infty}_{s,t} \right\rangle = 2\int_s^t (x^i(u) - x^i(s))\mathrm{d}x^i(u) = 2\int_s^t \underbrace{x^i(u)\dot{x}^i(u)}_{=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}u}(x^i(u))^2}\mathrm{d}u - 2x^i(s)x^i_{s,t}$$

$$= (x^i(t))^2 - (x^i(s))^2 - 2x^i(s)x^i(t) + 2(x^i(s))^2 = (x^i_{s,t})^2 = \left\langle \mathtt{i},\, \mathbb{x}^{<\infty}_{s,t} \right\rangle^2 = \left\langle \ell_1,\, \mathbb{x}^{<\infty}_{s,t} \right\rangle \left\langle \ell_2,\, \mathbb{x}^{<\infty}_{s,t} \right\rangle.$$

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  Note the redundancies in the signature!
- Given $p \in \mathbb{R}[x]$ (e.g., $p(x) = \lambda_0 + \lambda_1 x + \cdots + \lambda_n x^n$) and $\ell \in \mathcal{W}_d$, there is $p^{\sqcup\!\sqcup}(\ell) \in \mathcal{W}_d$, s.t.,

$$p\left(\left\langle \ell, \mathbb{X}^{<\infty}_{s,t} \right\rangle\right) = \left\langle p^{\sqcup\!\sqcup}(\ell), \mathbb{X}^{<\infty}_{s,t} \right\rangle, \quad p^{\sqcup\!\sqcup}(\ell) \coloneqq \lambda_0 \varnothing + \lambda_1 \ell + \cdots + \lambda_n \ell^{\sqcup\!\sqcup n} \in \mathcal{W}_d.$$

Polynomials in the signature are linear functionals in the signature.

Recall that signatures are invertible w.r.t. the tensor multiplication. Do they form a group?

Definition (Group-like elements)

$$G(\mathbb{R}^d) := \left\{ \mathbf{a} \in T((\mathbb{R}^d)) \,\middle|\, \forall \ell_1, \ell_2 \in \mathcal{W}_d : \langle \ell_1, \mathbf{a} \rangle \langle \ell_2, \mathbf{a} \rangle = \langle \ell_1 \sqcup\!\sqcup \ell_2, \mathbf{a} \rangle \right\}$$

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- From the shuffle-identity, for any smooth path $x : [s,t] \to \mathbb{R}^d$, $\mathbb{x}^{<\infty}_{s,t} \in G(\mathbb{R}^d)$.
- If $\mathbf{a} \in G(\mathbb{R}^d)$, then $\mathbf{a} = \mathbf{1} + \tilde{\mathbf{a}}$ (with $\langle \varnothing, \tilde{\mathbf{a}} \rangle = 0$), and $\mathbf{a}^{-1} = \sum_{k=0}^{\infty} (-1)^k \tilde{\mathbf{a}}^{\otimes k}$.

Recall that signatures are invertible w.r.t. the tensor multiplication. Do they form a group?

Definition (Group-like elements)

$$G(\mathbb{R}^d) := \left\{ \mathbf{a} \in T((\mathbb{R}^d)) \,\middle|\, \forall \ell_1, \ell_2 \in \mathcal{W}_d : \langle \ell_1, \mathbf{a} \rangle \langle \ell_2, \mathbf{a} \rangle = \langle \ell_1 \sqcup\!\sqcup \ell_2, \mathbf{a} \rangle \right\}$$

- From the shuffle-identity, for any smooth path $x : [s,t] \to \mathbb{R}^d$, $\mathbb{x}_{s,t}^{<\infty} \in G(\mathbb{R}^d)$.
- If $\mathbf{a} \in G(\mathbb{R}^d)$, then $\mathbf{a} = \mathbf{1} + \tilde{\mathbf{a}}$ (with $\langle \emptyset, \tilde{\mathbf{a}} \rangle = 0$), and $\mathbf{a}^{-1} = \sum_{k=0}^{\infty} (-1)^k \tilde{\mathbf{a}}^{\otimes k}$.
- We can also define a group $G^N(\mathbb{R}^d) \subset T^N(\mathbb{R}^d)$ by truncation. $G^N(\mathbb{R}^d)$ is a Lie group.

Define $\exp : T((\mathbb{R}^d)) \to T((\mathbb{R}^d))$ and $\log : \left\{ \mathbf{a} \in T((\mathbb{R}^d)) \,\middle|\, \langle \varnothing, \mathbf{a} \rangle = 1 \right\} \to T((\mathbb{R}^d))$ by

$$\exp(\mathbf{a}) := \mathbf{1} + \sum_{k=1}^{\infty} \frac{1}{k!} \mathbf{a}^{\otimes k}, \quad \log(\mathbf{a}) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \tilde{\mathbf{a}}^{\otimes k}, \qquad \text{with } \mathbf{a} = \mathbf{1} + \tilde{\mathbf{a}}.$$

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Lie algebra

$\mathfrak{g}(\mathbb{R}^d) := \log(G(\mathbb{R}^d))$ is a Lie algebra under the commutator $[\mathbf{a}, \mathbf{b}] := \mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a}$. In fact, it is the free Lie algebra generated by $e_1, \ldots, e_d$. Similarly, define $\mathfrak{g}^N(\mathbb{R}^d)$.

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- Note that $\exp : \mathfrak{g}(\mathbb{R}^d) \to G(\mathbb{R}^d)$ and $\log : G(\mathbb{R}^d) \to \mathfrak{g}(\mathbb{R}^d)$ are both bijective, and the same holds, mutatis mutandis, for the truncated versions $G^N(\mathbb{R}^d)$, $\mathfrak{g}^N(\mathbb{R}^d)$. Hence, $\mathfrak{g}^N(\mathbb{R}^d)$ is a global chart of the Lie group $G^N(\mathbb{R}^d)$.

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- $\dim \mathfrak{g}^N(\mathbb{R}^d)$ grows much slower than $\dim T^N(\mathbb{R}^d)$. E.g., for $d = 3$ and $N = 4$: $\dim T^N(\mathbb{R}^d) = 120$, $\dim \mathfrak{g}^N(\mathbb{R}^d) = 32$. Hence, the Lie algebra removes many redundancies (at the cost of the shuffle identity).

Definition (Log-signature)

Given a smooth path $x : [s, t] \to \mathbb{R}^d$, define the (truncated) log-signature by $\mathbb{l}^{<\infty}_{s,t} := \log(\mathbb{x}^{<\infty}_{s,t}) \in \mathfrak{g}(\mathbb{R}^d)$ – and similarly its truncated version $\mathbb{l}^{\leq N}_{s,t} \in \mathfrak{g}^N(\mathbb{R}^d)$.

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- By the definition of $\log$ applied to $\mathbb{x}^{\leq 2}_{s,t} = \mathbf{1} + x^i_{s,t} e_i + \mathbb{x}^{(i,j)}_{s,t} e_i \otimes e_j$, we get $\log \mathbb{x}^{\leq 2}_{s,t} = (\mathbb{x}^{\leq 2}_{s,t} - \mathbf{1}) - \frac{1}{2}(\mathbb{x}^{\leq 1}_{s,t} - \mathbf{1})^{\otimes 2} = x^i_{s,t} e_i + \left(\mathbb{x}^{(i,j)}_{s,t} - \frac{1}{2} x^i_{s,t} x^j_{s,t}\right) e_i \otimes e_j.$
- Note that $\mathbb{x}^{(i,j)}_{s,t} + \mathbb{x}^{(j,i)}_{s,t} = \int_{s<t_1<t_2<t} \mathrm{d}x^i(t_1)\mathrm{d}x^j(t_2) + \int_{s<t_2<t_1<t} \mathrm{d}x^i(t_1)\mathrm{d}x^j(t_2) = \int_s^t \int_s^t \mathrm{d}x^i(t_1)\mathrm{d}x^j(t_2) = x^i_{s,t} x^j_{s,t}$. Hence, $\mathbb{x}^{(i,i)}_{s,t} - \frac{1}{2}(x^i_{s,t})^2 = 0$, $\mathbb{x}^{(i,j)}_{s,t} - \frac{1}{2} x^i_{s,t} x^j_{s,t} = \frac{1}{2}(\mathbb{x}^{(i,j)}_{s,t} - \mathbb{x}^{(j,i)}_{s,t})$.
- In total: $\log \mathbb{x}^{\leq 2}_{s,t} = \sum_{i=1}^d x^i_{s,t} e_i + \sum_{1 \leq i < j \leq d} \frac{1}{2}(\mathbb{x}^{(i,j)}_{s,t} - \mathbb{x}^{(j,i)}_{s,t})[e_i, e_j] =: \sum_{i=1}^d x^i_{s,t} e_i + \sum_{1 \leq i < j \leq d} \mathbb{a}^{(i,j)}_{s,t}$.

Example: Signatures and areas of $x(t) = (\alpha\cosh(\theta_1 t) - \alpha,\ \cos(\theta_2 t))$, $d = 2$

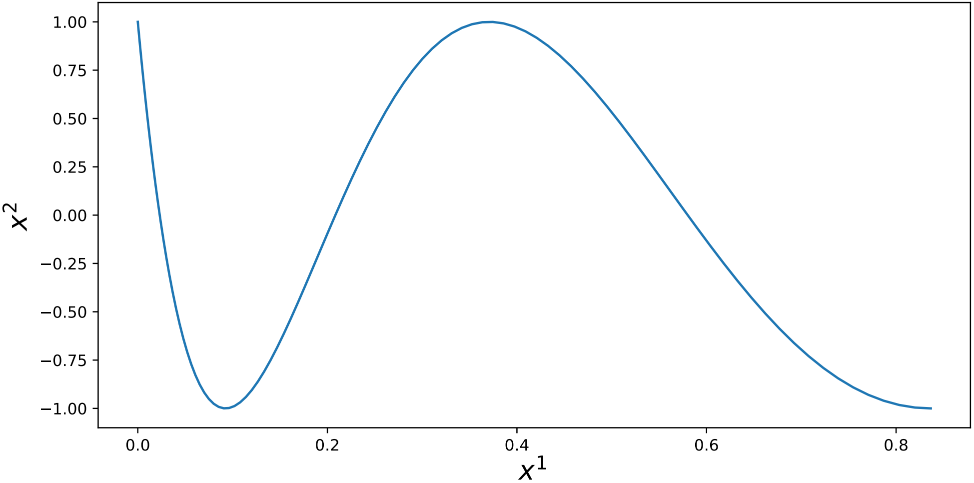

Figure: The path – up to re-parameterization.

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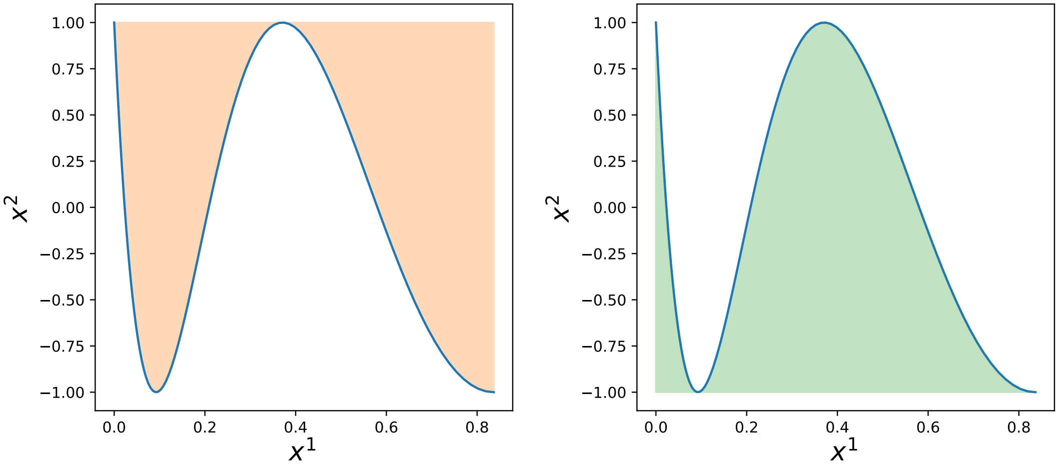

Figure: The shuffle identity $\mathbb{x}_{s,t}^{(1,2)} + \mathbb{x}_{s,t}^{(2,1)} = x_{s,t}^1 x_{s,t}^2$

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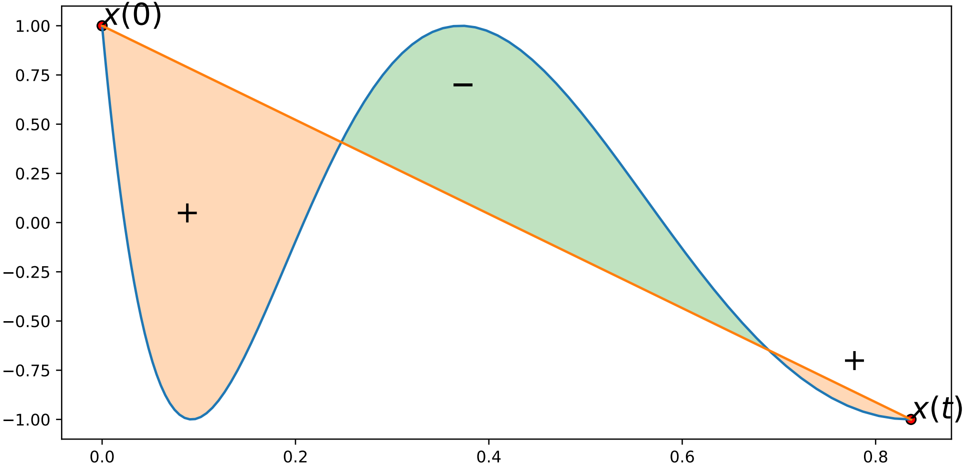

Figure: Interpretation of Lévy's area

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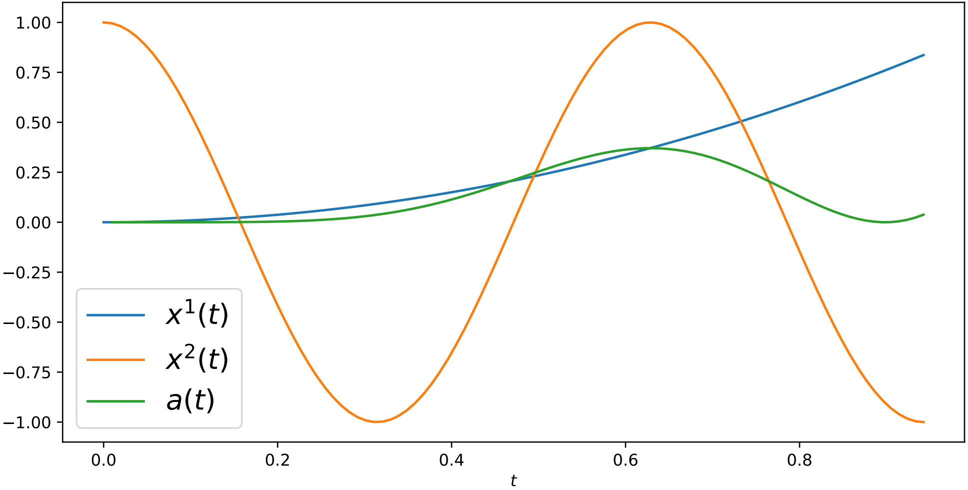

Figure: The path and the induced area path $t \mapsto \mathbb{a}_{0,t}$.

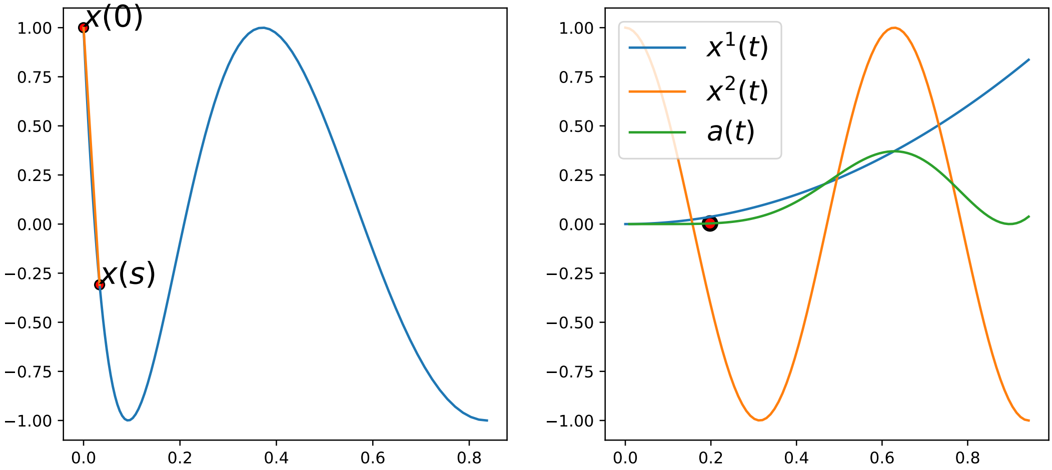

Figure: Construction of the induced area path $t \mapsto \mathbb{a}_{0,t}$.

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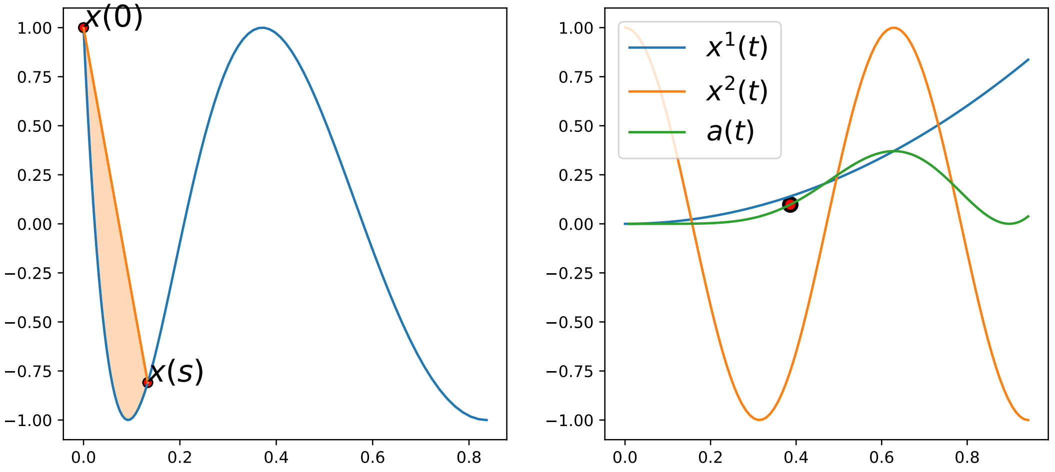

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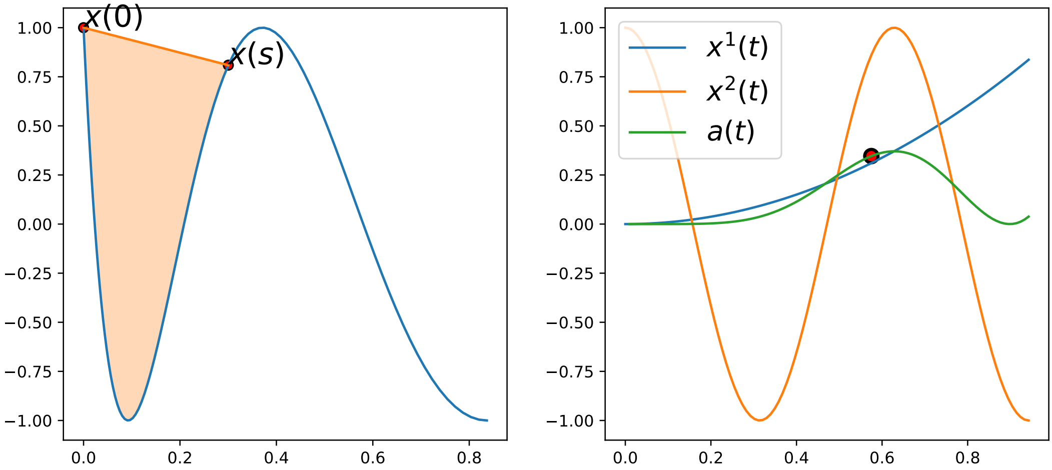

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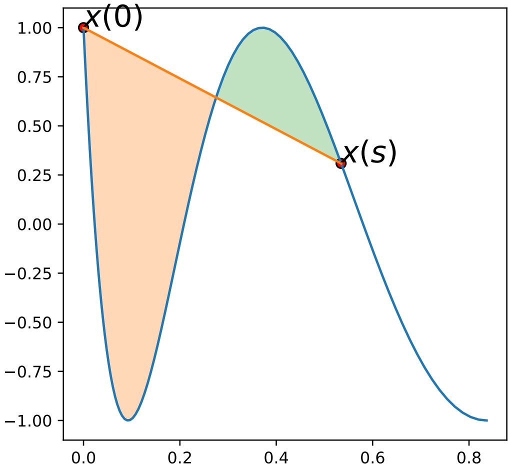

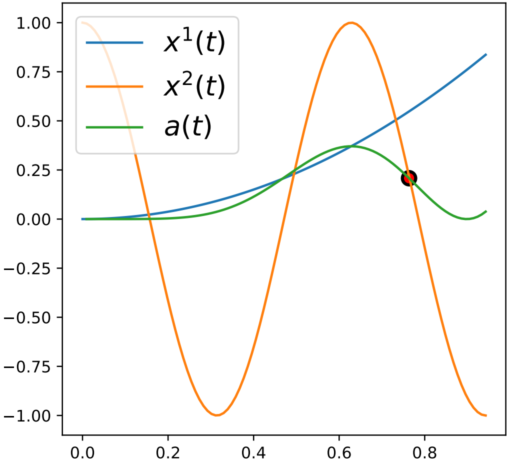

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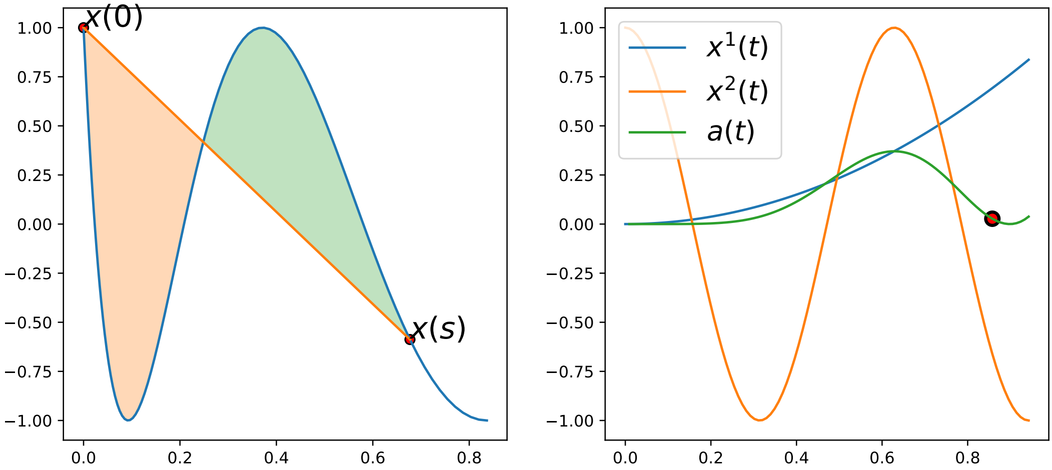

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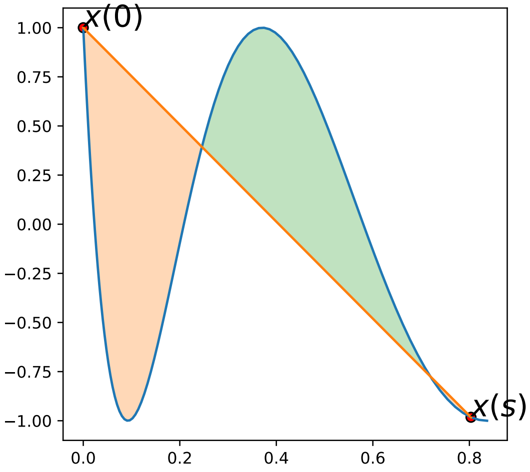

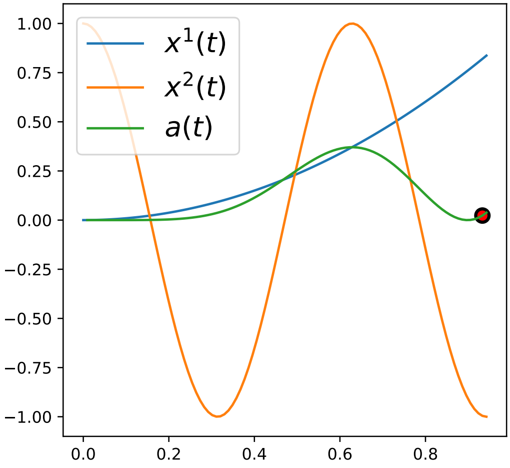

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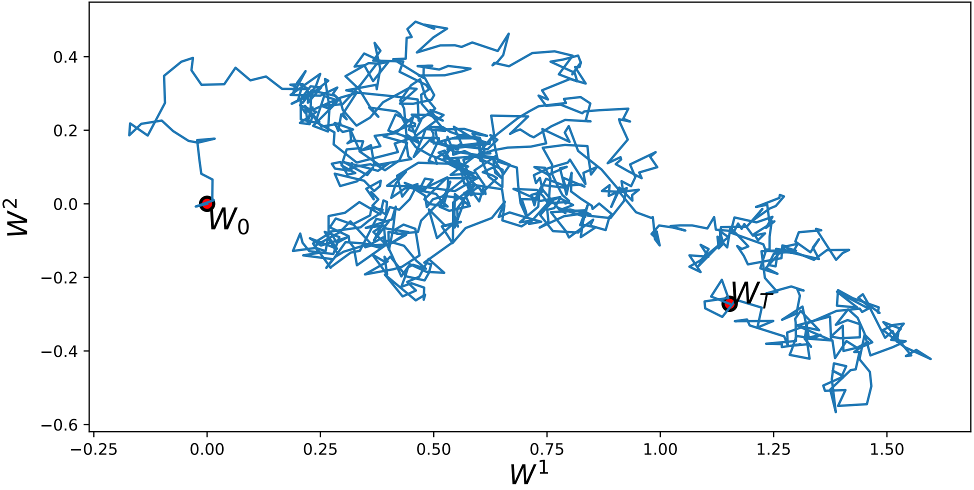

Figure: Path of a two-dimensional Brownian motion

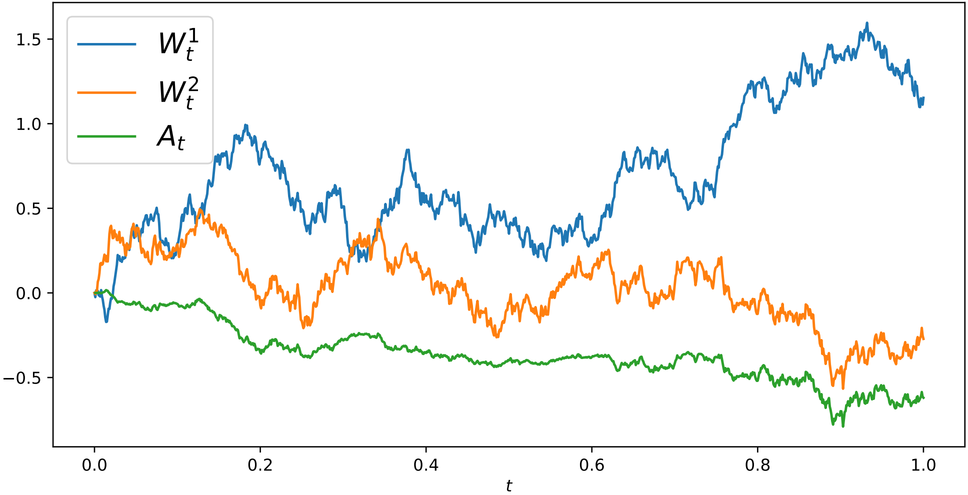

Figure: Path and area of a two-dimensional Brownian motion

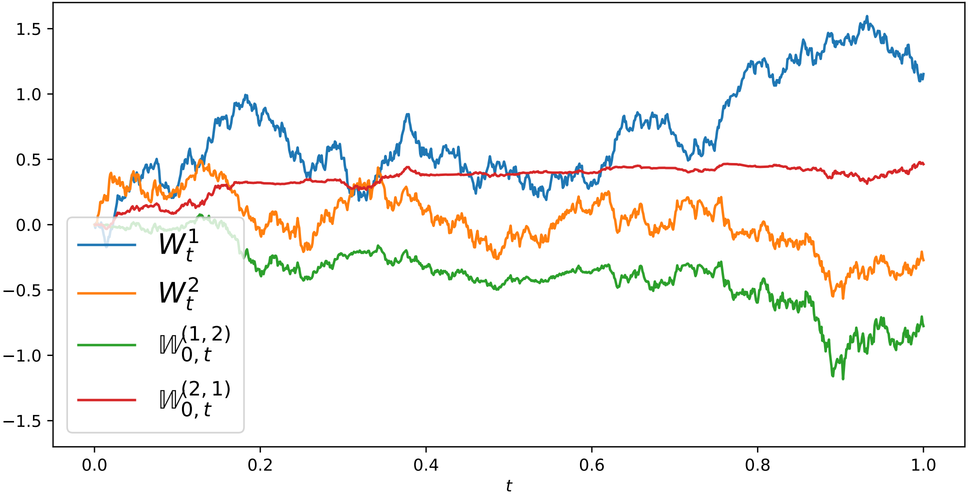

Figure: Path of W and non-trivial entries of $\mathbb{W}_{0,t}^{\leq 2}$ – note that $\mathbb{W}_{0,t}^{(i,i)} = \frac{1}{2}(W_{0,t}^i)^2$.

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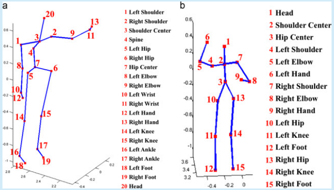
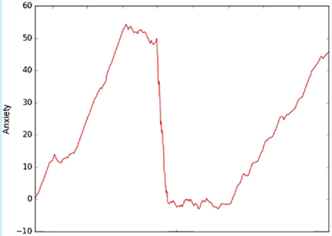
Examples [Terry Lyons and co-authors]

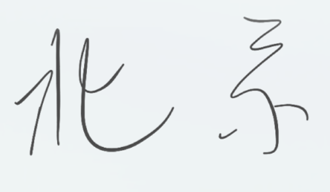
Human action recognotion



Psychiatric diagnosis



Chinese handwriting



- Time-extended path: Recall that the signature $\mathbb{x}_{s,t}^{<\infty}$ is invariant under re-parameterization. If this is not appropriate, extend x to $\overline{x}(u) := (u, x(u)) \in \mathbb{R}^{d+1}$. Its signature $\overline{\mathbb{x}}_{s,t}^{<\infty}$ effectively respects the given parameterization.

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Modern trends

- Neural (rough) DEs.
- Signature kernel methods

- K.-T. Chen. Iterated integrals and exponential homomorphisms, Proceedings of the London Mathematical Society 3(1):502–512, 1954.
- I. Chevyrev, A. Kormilitzin. A primer on the signature method in machine learning, arXiv:1603.03788, 2016.
- M. Fliess. Fonctionnelles causales non linéaires et indéterminées non commutatives, Bulletin de la société mathématique de France 109:3–40, 1981.
- P.K. Friz, B. Nicolas. Multidimensional stochastic processes as rough paths: theory and applications, Vol. 120. Cambridge University Press, 2010.

Figure: Kuo-Tsai Chen (1923–1987)

Can we solve $\mathrm{d}y(t) = V(y(t))\mathrm{d}x(t)$ for a non-smooth path $x : [0, T] \to \mathbb{R}^d$ – e.g., α-Hölder?

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Example

- Let $x_n(t) := \left(\sin(n^2 t)/n, \cos(n^2 t)/n\right)$, $t \in [0, 2\pi]$, with limit $x(t) \equiv 0$, and the area

$$z_n(t) := \frac{1}{2}\int_0^t x_n^1(s)\mathrm{d}x_n^2(s) - \frac{1}{2}\int_0^t x_n^2(s)\mathrm{d}x_n^1(s)$$

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- Note that $y_n(t) := (x_n^1(t), x_n^2(t), z_n(t))$ solves controlled DE with $V(y) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2}y^2 & -\frac{1}{2}y^1 \end{pmatrix}$.

$$dy(t) = V(y(t))dx(t), \ t \in [0, T], \ y(0) = y_0, \ x : [0, T] \to \mathbb{R}^d, \ 0 = t_0 < \cdots < t_n = T.$$

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Case: smooth path x. If x is smooth, we have $\left|x_{t_i,t_{i+1}}\right| = \mathcal{O}(|t_{i+1} - t_i|)$. By Taylor,

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Remark: (Young '30s) $\int_0^T f(s)\mathrm{d}g(s)$ well-defined for f α-Hölder, g β-Hölder iff $\alpha + \beta > 1$.

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Now consider x to be α-Hölder with $\frac{1}{3} < \alpha \leq \frac{1}{2}$. By the previous calculation, the Euler scheme diverges. Recall the formal second order expansion:

$$y(t_{i+1}) + V(y(t_i))x_{t_i,t_{i+1}} + DV(y(t_i))V(y(t_i))\mathbb{x}^{=2}_{t_i,t_{i+1}} + \text{H.O.T.}_i.$$

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Key observation

Assume that we could define $\mathbb{x}^{=2}_{t_i,t_{i+1}} = \left(\int_{t_i}^{t_{i+1}} x^j_{t_i,s}\mathrm{d}x^k(s)\right)_{j,k=1,\dots,d}$. Then we would expect

$$\left|x_{t_i,t_{i+1}}\right| = O(|t_{i+1}-t_i|^{\alpha}), \quad \left|\mathbb{x}^{=2}_{t_i,t_{i+1}}\right| = O(|t_{i+1}-t_i|^{2\alpha}), \quad |\text{H.O.T.}_i| = O(|t_{i+1}-t_i|^{3\alpha}) = o(|t_{i+1}-t_i|).$$

Hence, we expect convergence of the extended Euler scheme

$$\bar{y}_{i+1} = \bar{y}_i + V(\bar{y}_i)x_{t_i,t_{i+1}} + DV(\bar{y}_i)V(\bar{y}_i)\mathbb{x}^{=2}_{t_i,t_{i+1}}.$$

Definition (α-Hölder rough paths)

Let $\frac{1}{3} < \alpha \leq \frac{1}{2}$. An α-Hölder rough path on $\mathbb{R}^d$ is a pair $\mathbf{x} = (x, \mathbb{x})$, $x : [0, T] \to \mathbb{R}^d$, $\mathbb{x} : [0, T]^2 \to \mathbb{R}^d \otimes \mathbb{R}^d$, continuous, such that Chen's identity (truncated to $N = 2$) holds and

$$\sup_{s \neq t} \frac{\left| x_{s,t} \right|}{|t - s|^{\alpha}} < \infty, \quad \sup_{s \neq t} \frac{\left| \mathbb{x}_{s,t} \right|}{|t - s|^{2\alpha}} < \infty.$$

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- Every α-Hölder path can be extended to an α-Hölder rough path, but the extension is generally not unique. (N.b.: If x is smooth, there is a canonical choice.)
- The theory of rough paths was developed by Terry Lyons starting from 1994. Important re-formulations and generalizations were due to Massimiliano Gubinelli (controlled rough paths) and Martin Hairer (regularity structures).

Universal limit theorem

Given an α-Hölder rough path $\mathbf{x}$, and $V \in C^\gamma$ for $\gamma \geq 1/\alpha$. Then there is a unique solution of the rough differential equation

$$\mathrm{d}y(t) = V(y(t))\mathrm{d}\mathbf{x}(t), \quad y(0) = y_0.$$

The map $(y_0, V, \mathbf{x}) \to y$ is locally Lipschitz continuous – w.r.t. appropriate topologies.

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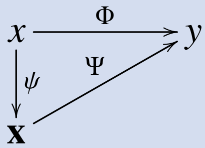
- As the signature solves the RDE $\mathrm{d}\mathbb{x}^{<\infty}_{s,t} = \mathbb{x}^{<\infty}_{s,t} \otimes \mathrm{d}\mathbf{x}(t)$, $\mathbb{x}^{<\infty}_{s,s} = \mathbf{1}$, this implies that every rough path has a uniquely defined signature.
- The solution y depends on the rough path $\mathbf{x}$, i.e., the choice of extension of x.

Rough path principle



Given a d-dimensional Brownian motion W. We can construct iterated integrals (in an L^2 or almost sure sense) as follows

- $$\mathbb{W}^{(i,j),\text{Ito}}_{s,t} := \int_s^t W^i_{s,u} \mathrm{d}W^j_u, \quad \mathbb{W}^{=2,\text{Ito}}_{s,t} := \sum_{1\le i,j\le d} \mathbb{W}^{(i,j),\text{Ito}}_{s,t} e_i \otimes e_j,$$
- $$\mathbb{W}^{(i,j),\text{Strat}}_{s,t} := \int_s^t W^i_{s,u} \circ \mathrm{d}W^j_u, \quad \mathbb{W}^{=2,\text{Strat}}_{s,t} := \sum_{1\le i,j\le d} \mathbb{W}^{(i,j),\text{Strat}}_{s,t} e_i \otimes e_j.$$

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- Note that $\mathbb{W}^{<\infty,\text{Strat}}$ satisfies the shuffle identity, but $\mathbb{W}^{<\infty,\text{Ito}}$ does not.

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- The signature $\mathbb{x}_{s,t}^{<\infty}$ of a geometric rough path $\mathbf{x} \in \mathscr{C}_g^\alpha$ satisfies the shuffle identity. Symbolically,

$$\forall \mathbf{x} \in \mathscr{C}_g^\alpha([0,T];\mathbb{R}^d),\ \forall 0 \le s \le t \le T : \ \mathbb{x}_{s,t}^{<\infty} \in G(\mathbb{R}^d).$$

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Figure: Terry Lyons

W.l.o.g., all paths start at 0, i.e., $x(0) = 0$.

- Let $\Omega_1 := \mathscr{C}^{1-\text{var}}([0,T];V)$ denote the space of bounded variation functions taking values in a (finite-dimensional) Banach space V with the norm $\|x\|_{1-\text{var}} := |x(0)| + |x|_{1-\text{var}}$, where

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Theorem

Let $A := \{ f_\ell \mid \ell \in \mathcal{W}_{1+d} \}$ *where for any* $\ell \in \mathcal{W}_{1+d}$ *we set*

$$f_\ell : \widehat{\Omega}_1 \to \mathbb{R}, \quad \widehat{x} \mapsto \left\langle \ell, \widehat{\mathbb{x}^{<\infty}_{0,T}} \right\rangle.$$

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The proof is based on the classical Stone – Weierstrass theorem. We give a sufficient version below:

Theorem (Stone – Weierstrass)

Let X *be a compact metric space and consider a subalgebra* $A \subset C(X; \mathbb{R})$ *that is point-separating and vanishes nowhere. Then* $A \subset C(X; \mathbb{R})$ *is dense w.r.t. uniform convergence.*

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$$\Omega_p := \mathscr{C}_g^{1/p}([0,T];\mathbb{R}^d), \quad \widehat{\Omega}_p := \left\{ \mathbf{x} = (x, \mathbb{x}) \in \mathscr{C}_g^{1/p}([0,T];\mathbb{R}^{1+d}) \,\middle|\, \forall t \in [0,T] : \; x^0(t) = t \right\}.$$

- We can replace $\widehat{\Omega}_1$ by $\mathscr{P}_1$, the set of bounded variation paths modulo re-parameterization and tree-like excursion.
- We can immediately generalize the theorem to the rough setting, i.e., by replacing Ω_1 and $\widehat{\Omega}_1$ by their rough analogues for $p > 1$:

$$\Omega_p := \mathscr{C}_g^{1/p}([0,T];\mathbb{R}^d), \quad \widehat{\Omega}_p := \left\{ \mathbf{x} = (x, \mathbb{x}) \in \mathscr{C}_g^{1/p}([0,T];\mathbb{R}^{1+d}) \,\middle|\, \forall t \in [0,T] : \ x^0(t) = t \right\}.$$

- Unlike $\mathscr{C}^{1/p}([0,T];\mathbb{R}^d)$, $\mathscr{C}_g^{1/p}([0,T];\mathbb{R}^d)$ is separable, hence a Polish space. Any rough process defined as a random variable taking values in Ω_p or $\widehat{\Omega}_p$, respectively, is tight.

Corollary

Given a rough process $\widehat{\mathbf{X}}$ taking values in $\widehat{\Omega}_p$, $p > 1$. Then for any $f \in C\left(\widehat{\Omega}_p;\mathbb{R}\right)$ and $\epsilon > 0$ there is $\ell \in \mathcal{W}_{1+d}$ s.t.

$$P\left(\left|f(\widehat{\mathbf{X}}) - \left\langle \ell, \widehat{\mathbb{X}}^{<\infty}_{0,T} \right\rangle\right| > \epsilon\right) < \epsilon.$$

Theorem (Stone – Weierstrass theorem; Giles'71)

Let X be a ~~compact~~ metric space and consider a subalgebra $A \subset C_b(X;\mathbb{R})$ that is point-separating and vanishes nowhere. Then $A \subset C_b(X;\mathbb{R})$ is dense w.r.t. the strict topology.

- The strict topology on $C_b(X;\mathbb{R})$ is the topology generated by the seminorms $p_\psi(f) := \sup_{x \in X} |f(x)\psi(x)|$, $f \in C_b(X;\mathbb{R})$, indexed by the functions $\psi : X \to \mathbb{R}$ vanishing at infinity.

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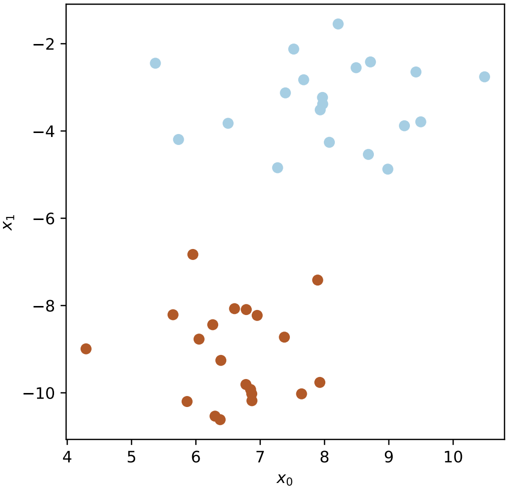
- Replace the (unbounded) functions $\widehat{x} \mapsto \left\langle \ell, \widehat{\mathbb{x}}^{<\infty}_{0,T} \right\rangle$ by the bounded functions $\widehat{x} \mapsto \left\langle \ell, \Lambda(\widehat{\mathbb{x}}^{<\infty}_{0,T}) \right\rangle$ for a tensor normalization $\Lambda : T((\mathbb{R}^d)) \to T((\mathbb{R}^d))$.
- Tensor normalizations are continuous, injective maps Λ s.t. $\Lambda(\mathbf{a})$ is in a bounded ball in $T((\mathbb{R}^d))$ and $\Lambda(\mathbf{a}) = \delta_{\lambda(\mathbf{a})}\mathbf{a}$ for some $\lambda : T((\mathbb{R}^d)) \to \mathbb{R}$.

- Consider data $x_i \in E$ for a (finite-dimensional) space E, with labels $y_i \in \{-1, +1\}$, $i = 1, \dots, M$.
- Classify data points by a separating hyperplane, i.e., find $w \in E$ and $b \in \mathbb{R}$ s.t. for all $i = 1, \dots, M$:

$$y_i = +1 \iff \langle w, x_i \rangle_E - b > 0,$$
$$y_i = -1 \iff \langle w, x_i \rangle_E - b < 0.$$



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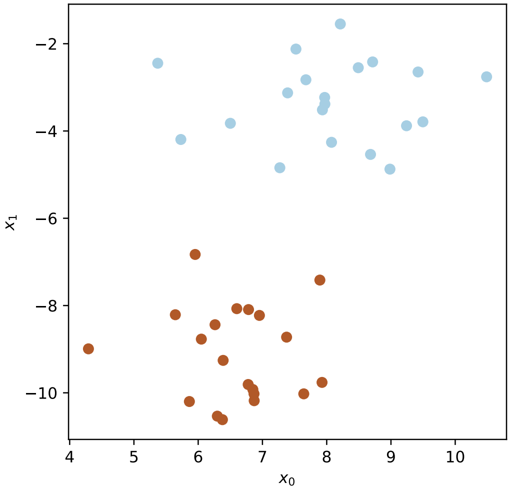
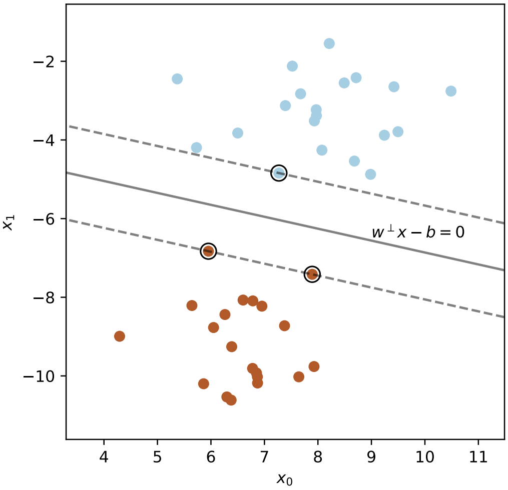
- If at all possible, there will be infinitely many solutions. Hence, we try to find the best solution.



Solution

$$\min_{w \in E,\, b \in \mathbb{R}} \frac{1}{2} \|w\|_E^2 \quad \text{subject to}$$

$$\forall i \in \{1, \ldots, M\}: \; y_i (\langle w, x_i \rangle_E - b) \geq 1.$$



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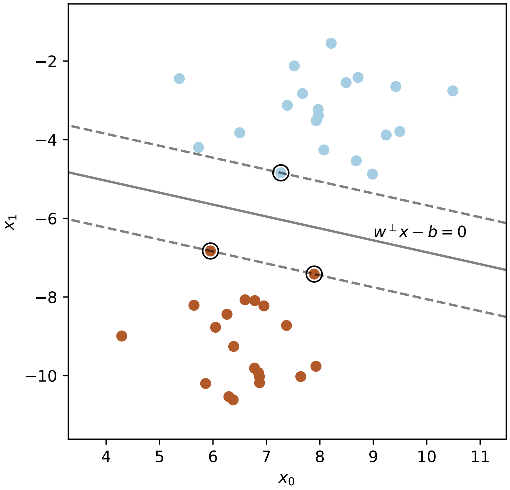
- What if separation by hyperplanes is not possible, or data lives in a non-linear space $\mathcal{X}$?
- Lift data $x_i \mapsto \Phi(x_i)$ using a non-linear feature map $\Phi : \mathcal{X} \to \mathcal{H}$ for some (infinite-dimensional) Hilbert space $\mathcal{H}$.
- Which Φ? Evaluation very expensive!?



Definition

A reproducing kernel Hilbert space (RKHS) is a Hilbert space $\mathcal{H}$ of functions $f : \mathcal{X} \to \mathbb{R}$ s.t. for all $x \in \mathcal{X}$, the evaluation functional $\mathrm{ev}_x : \mathcal{H} \to \mathbb{R}$, $f \mapsto f(x)$ is continuous.

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- By Riesz representation, for every $x \in \mathcal{X}$ we can find $k_x \in \mathcal{H}$ such that

$$\forall f \in \mathcal{H} : \ \mathrm{ev}_x(f) = \langle k_x, f \rangle_{\mathcal{H}} \, .$$

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1. By the analogue properties of $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, k is symmetric and positive definite, i.e., $\forall x_1, \dots, x_k \in \mathcal{X}$, the matrix $(k(x_i, x_j)) \in \mathbb{R}^{k \times k}$ is positive definite.
2. $k_x(y) = \mathrm{ev}_y(k_x) = \left\langle k_y, k_x \right\rangle_{\mathcal{H}} = k(x, y)$, i.e., for any $x \in \mathcal{X}$, $k_x = k(x, \cdot)$.

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3. Conversely, given a symmetric, positive definite kernel $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, we obtain a RKHS as completion of $\widetilde{H} := \langle \{ k(x, \cdot) \mid x \in \mathcal{X} \} \rangle$ with $\langle k(x, \cdot), k(y, \cdot) \rangle_{\widetilde{\mathcal{H}}} := k(x,y)$.

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- By the representer theorem, $w \in \langle \{ k(x_i, \cdot) \mid i = 1, \dots, M \} \rangle$, i.e.,

$$\exists \alpha \in \mathbb{R}^M : \; w = \sum_{i=1}^{M} \alpha_i k(x_i, \cdot), \text{ hence } \|w\|_{\mathcal{H}}^2 = \sum_{i=1}^{M} \alpha^\top K \alpha, K := (k(x_i, x_j))_{i,j=1}^{M} \in \mathbb{R}^{M \times M}.$$

- Similarly, $\langle w, \Phi(x_i) \rangle_{\mathcal{H}} = \sum_{j=1}^{M} \alpha_j \left\langle k(x_j, \cdot), k(x_i, \cdot) \right\rangle_{\mathcal{H}} = (\alpha^\top K)_i$.

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- Need evaluations of the kernel k (for the Gram matrix K), but not of Φ – kernel trick.

Let $\mathcal{X}_1 := \left\{ x \in \mathscr{C}^{1-\mathrm{var}}([0,T];\mathbb{R}^d) \,\middle|\, T > 0,\ x(0) = 0 \right\}$ – and similarly $\widehat{\mathcal{X}}_1$.

Goal: Define an appropriate kernel for paths / time series.

Definition

Given $x, y \in \mathcal{X}_1$ defined on $[0,t], [0,s]$, respectively. We define

$$\mathrm{k}_{\mathrm{sig}}(x,y) := \left\langle \mathbb{x}^{<\infty}_{0,t}, \mathbb{y}^{<\infty}_{0,s} \right\rangle := \sum_{n=0}^{\infty} \sum_{\alpha \in \{1,\dots,d\}^n} \mathbb{x}^{\alpha}_{0,t} \mathbb{y}^{\alpha}_{0,s}.$$

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- It is easy to see that $\left| \mathbb{x}_{[0,t]}^{=n} \right| \le \frac{\|x\|_{1-\mathrm{var}}^n}{n!}$, therefore the sum is finite.
- The definition can easily be extended to rough paths or time series – e.g., by piecewise-linear interpolation.
- Extension: For a kernel $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, first lift $t \mapsto x(t) \to \kappa_x := t \mapsto \kappa(x(t), \cdot) \in \mathcal{H}$, then compute the signature kernel of the lifted path $\mathrm{k}_{\mathrm{sig}}(\kappa_x, \kappa_y)$.

Direct computation is impossible, due to the exponential growth of the signature – recall that $\mathbb{x}^{=n} \in (\mathbb{R}^d)^{\otimes n}$, i.e., has d^n terms. However, a recursive construction exists – comparable to the Horner scheme for polynomials. Even more powerful:

Theorem [Salvi et al., '21]

Assume that $x, y \in C^1$, and let $K_{x,y}(u, v) := \mathrm{k}_{\mathrm{sig}}\left(x|_{[0,u]}, y|_{[0,v]}\right)$ for $u \in [0, t]$, $v \in [0, s]$. Then $K_{x,y}$ solves the PDE

$$\frac{\partial^2}{\partial u \partial v} K_{x,y}(u, v) = \langle \dot{x}(u), \dot{y}(v) \rangle K_{x,y}(u, v), \quad K_{x,y}(0, \cdot) = K_{x,y}(\cdot, 0) = 1.$$

$$\mathrm{MMD}_{\mathsf{sig}}(\mu,\nu) := \Big[\int_{\mathcal{X}_1\times\mathcal{X}_1} \mathsf{k}_{\mathsf{sig}}(x,x')\mu(\mathrm{d}x)\mu(\mathrm{d}x') + \int_{\mathcal{X}_1\times\mathcal{X}_1} \mathsf{k}_{\mathsf{sig}}(y,y')\nu(\mathrm{d}y)\nu(\mathrm{d}y') - 2\int_{\mathcal{X}_1\times\mathcal{X}_1} \mathsf{k}_{\mathsf{sig}}(x,y)\mu(\mathrm{d}x)\nu(\mathrm{d}y)\Big]^{1/2}$$

- Given $\mathcal{K} \subset \widehat{\mathcal{X}_1}$ compact, then $\mathrm{MMD}_{\mathsf{sig}}$ is characteristic for $\mathcal{P}_1(\mathcal{K})$, the probability measures supported on $\mathcal{K}$, i.e., $\mathrm{MMD}_{\mathsf{sig}}(\mu,\nu) = 0 \iff \mu = \nu$.

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- Given $\mathcal{K} \subset \widehat{\mathcal{X}_1}$ compact, then $\mathrm{MMD}_{\mathsf{sig}}$ is characteristic for $\mathcal{P}_1(\mathcal{K})$, the probability measures supported on $\mathcal{K}$, i.e., $\mathrm{MMD}_{\mathsf{sig}}(\mu,\nu) = 0 \iff \mu = \nu$.
- In the compact case, $\mathrm{MMD}_{\mathsf{sig}}$ is a metric for weak convergence.
- For $\mathcal{P}_1(\widehat{\mathcal{X}_1})$ we obtain a metric by switching to normalized signatures, as discussed earlier. However, convergence under $\mathrm{MMD}_{\mathsf{sig}}$ does not imply weak convergence.

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Figure: Vladimir Vapnik

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Outline

1 Path signatures

2 Rough Paths

3 Universality and the signature kernel

4 **Signature based representations for optimal stopping**

Setting

Given a d-dimensional stochastic process $(X_t)_{t\in[0,T]}$ controlled by α. Goal: maximize some reward function.

Markovian case: If X is a Markov process, the optimal control satisfies $\alpha_t^* = \alpha^*(t, X_t)$.

Popular methods include:

- Solving the (deterministic) Hamilton–Jacobi–Belman PDE for the value function.
- Approximate α^* in some parametric class of functions on $\mathbb{R}^d$ and optimize the reward.
- Least squares Monte Carlo, involving computations of conditional expectations $E[V_{t+\Delta t} \mid X_t]$.

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Non-Markovian case: Now we can only expect α_t^* to be $\mathcal{F}_t$-measurable, i.e., $\alpha_t^* = \alpha^*(t, (X_s)_{s\le t})$. For all methods above, we are left with approximations in spaces of functions of paths.

Following [Kalsi, Lyons, Perez Arribas '20], a general recipe for solving stochastic optimal control problems using path signatures can be described as follows:

1. Assume that controls α_t are continuous functions $\phi(\widehat{X}|_{[0,t]})$ of the path and, hence, of the signature $\theta(\widehat{\mathbb{X}}^{<\infty}_{0,t})$ – and similarly for the loss function $L_\theta(\widehat{\mathbb{X}}^{<\infty}_{0,T})$.

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2. As continuous functions, $\alpha_t = \theta(\widehat{\mathbb{X}}^{<\infty}_{0,t}) \approx \left\langle \ell_\theta, \widehat{\mathbb{X}}^{<\infty}_{0,t} \right\rangle$, $L_\theta(\widehat{\mathbb{X}}^{<\infty}_{0,T}) \approx \left\langle f_L(\ell_\theta), \widehat{\mathbb{X}}^{<\infty}_{0,T} \right\rangle$ for some $\ell_\theta, f_L(\ell_\theta) \in \mathcal{W}_d$ – by universality.

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No convergence result known so far, but pathwise density for steps 1 + 2 with high probability is proved in [Kalsi, Lyons, Perez Arribas '20]. Problem: discontinuity of (optimal) controls.

Optimal stopping problem

Given a stochastic reward process $(Y_t)_{t\in[0,T]}$ adapted to a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by a d-dimensional stochastic process $(X_t)_{t\in[0,T]}$. Let $\mathcal{S}$ denote the set of $(\mathcal{F}_t)_{t\in[0,T]}$–stopping times. Compute $\sup_{\tau\in\mathcal{S}} \mathbb{E}\left[Y_\tau\right]$.

Optimal stopping problem

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- Optimal stopping times are generally hitting times of sets, hence discontinuous functions on path-space.

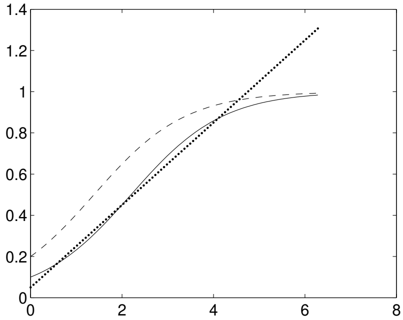

Figure: Discontinuity of hitting times

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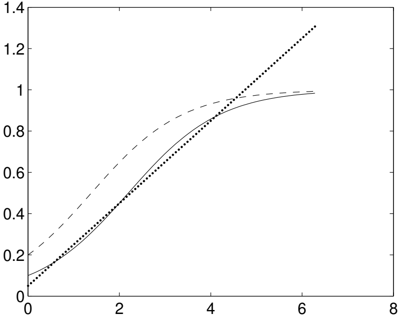
- Optimal stopping times are generally hitting times of sets, hence discontinuous functions on path-space.
- **Example:** X models a stock price – possibly with additional factors such as stochastic volatilities – and $Y_t = h(X_t)$ for some payoff function h.
- **Example:** $X = Y = W^H$ . . . fractional Brownian motion



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[Becker, Cheredito, Jentzen '19] consider the optimal stopping problem for fractional Brownian motion. In the general setting, their strategy is as follows:

1. Fix a time-grid $0 = t_0 < \cdots < t_J = T$ and define a (discrete time) $(J+1)d$-dimensional Markov process $(Z_j)_{j=0}^J$ by

$$\begin{aligned} Z_0 &:= (X_{t_0}, 0, \ldots, 0), \\ Z_1 &:= (X_{t_0}, X_{t_1}, 0, \ldots, 0), \\ Z_2 &:= (X_{t_0}, X_{t_1}, X_{t_2}, 0, \ldots, 0), \\ &\vdots \end{aligned}$$

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2. Solve the discrete-time Markovian optimal stopping problem. [Becker, Cheredito, Jentzen '19] use deep neural networks to approximate stopping decisions $f_j(Z_j) \approx \mathrm{DNN}_j(Z_j; \theta)$ – "stop at time t_j unless stopped earlier".

How can we construct stopping times and adapted processes using rough paths?

Stopped rough paths

Let $\widehat{\Omega}_t^p := \left\{ \mathbf{x} \in \mathscr{C}_g^{1/p}([0,t];\mathbb{R}^{1+d}) \;\middle|\; x^1(s) = s \right\}$. The space of stopped rough paths is defined as $\Lambda_T := \bigcup_{t\in[0,T]} \widehat{\Omega}_t^p$.

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Rough stochastic processes

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a rough stochastic process is a random variable $\widehat{\mathbf{X}}$ taking values in $\widehat{\Omega}_T^p$. We further define the natural filtration generated by $\widehat{\mathbf{X}}$, i.e., $\mathcal{F}_t := \sigma\left(\mathbf{X}_{0,s} : 0 \le s \le t\right)$.

Given $\ell \in \mathcal{W}_{1+d}$, define a signature stopping rule $\tau_\ell \in \mathcal{S}$ as

$$\tau_\ell := \inf\left\{ t \in [0,T] \,\middle|\, \left\langle \ell, \widehat{\mathbb{X}}^{<\infty}_{0,t} \right\rangle \geq 1 \right\}.$$

Note that τ_ℓ is the first hitting time of a hyperplane in $T((\mathbb{R}^{1+d}))$.

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Theorem

Given an $(\mathcal{F}_t)_{t\in[0,T]}$–adapted continuous reward process $(Y_t)_{t\in[0,T]}$ with $\mathbb{E}\,\|Y\|_\infty < \infty$, then

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- While optimal stopping times $\tau^* \in \mathcal{S}$ typically exist, we do not expect an optimizer $\ell^* \in \mathcal{W}_{1+d}$ to exist.

Given $\theta \in C(\Lambda_T, \mathbb{R})$ define a continuous stopping rule by

$$\tau_\theta := \inf\left\{ t \in [0,T] \,\middle|\, \int_0^t \theta(\widehat{\mathbf{X}}|_{[0,s]})^2 \mathrm{d}s \geq 1 \right\}.$$

Lemma

$$\sup_{\theta \in C(\Lambda_T, \mathbb{R})} \mathbb{E}\left[Y_{\tau_\theta \wedge T}\right] = \sup_{\tau \in \mathcal{S}} \mathbb{E}\left[Y_{\tau \wedge T}\right]$$

Proof of the Lemma is based on approximation of measurable by continuous functions.

- If a continuous stopping rule τ_θ was continuous as a function of the signature, we could approximate it by signature stopping rules:

$$\inf\left\{ t \in [0,T] \,\middle|\, \int_0^t \theta(\widehat{\mathbf{X}}|_{[0,s]})^2 \mathrm{d}s \geq 1 \right\} \approx \inf\left\{ t \in [0,T] \,\middle|\, \left\langle \ell, \widehat{\mathbb{X}}^{<\infty}_{0,t} \right\rangle \geq 1 \right\}.$$

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- **Randomization:** Replace the fixed level 1 above by an (independent) random level Z.
- Interpretation: If $Z \sim \mathrm{Exp}(1)$, stop at the first jump time of a pure jump process with intensity $\theta(\widehat{\mathbf{X}}|_{[0,s]})^2$.

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Let $Z \geq 0$ be a r.v. independent of $\widehat{\mathbb{X}}$ with (smooth) c.d.f. F_Z.

$$\tau^r_\theta := \inf\left\{ t \in [0,T] \,\middle|\, \int_0^t \theta\left(\widehat{\mathbb{X}}|_{[0,s]}\right)^2 \mathrm{d}s \geq Z \right\}, \quad \tau^r_\ell := \inf\left\{ t \in [0,T] \,\middle|\, \int_0^t \left\langle \ell, \widehat{\mathbb{X}}^{<\infty}_{0,t} \right\rangle^2 \mathrm{d}s \geq Z \right\}.$$

Lemma

$$\sup_{\theta \in C(\Lambda_T, \mathbb{R})} \mathbb{E}\left[Y_{\tau^r_\theta \wedge T}\right] = \sup_{\theta \in C(\Lambda_T, \mathbb{R})} \mathbb{E}\left[Y_{\tau_\theta \wedge T}\right], \quad \sup_{\ell \in \mathcal{W}_{1+d}} \mathbb{E}\left[Y_{\tau^r_\ell \wedge T}\right] = \sup_{\ell \in \mathcal{W}_{1+d}} \mathbb{E}\left[Y_{\tau_\ell \wedge T}\right].$$

Proof: Formal proof by dominated convergence. Informally: The buyer of an American option may very well randomize her exercise decision.

Lemma

$$\sup_{\theta \in C(\Lambda_T, \mathbb{R})} \mathbb{E}\left[Y_{\tau_\theta^r \wedge T}\right] = \sup_{\theta \in C(\Lambda_T, \mathbb{R})} \mathbb{E}\left[Y_{\tau_\theta \wedge T}\right], \quad \sup_{\ell \in \mathcal{W}_{1+d}} \mathbb{E}\left[Y_{\tau_\ell^r \wedge T}\right] = \sup_{\ell \in \mathcal{W}_{1+d}} \mathbb{E}\left[Y_{\tau_\ell \wedge T}\right].$$

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Lemma (Regularization by randomization)

Let $\widetilde{F}(t) := F_Z\left(\int_0^t \theta\left(\widehat{\mathbf{X}}|_{[0,s]}\right) \mathrm{d}s\right)$, then $\mathbb{E}\left[Y_{\tau_\theta^r \wedge T} \mid \widehat{\mathbf{X}}\right] = \int_0^T Y_t \mathrm{d}\widetilde{F}(t) + Y_T(1 - \widetilde{F}(T))$.

- Note that the R.H.S. is a smooth function of $\widehat{\mathbf{X}}$.

Lemma

For every $\varepsilon > 0$ there is a compact set $\mathcal{K} \subset \widehat{\Omega}_T^p$ s.t. $\mathbb{P}(\mathbf{X} \in \mathcal{K}) > 1 - \varepsilon$ and for every $\theta \in C(\Lambda_T, \mathbb{R})$ there is a sequence $\ell_n \in \mathcal{W}_{1+d}$ s.t.

$$\sup_{\mathbf{x} \in \mathcal{K};\ t \in [0,T]} \left| \theta(\widehat{\mathbf{x}}|_{[0,t]}) - \left\langle \ell_n, \mathbb{x}_{0,t}^{<\infty} \right\rangle \right| \xrightarrow{n \to \infty} 0.$$

The above Stone–Weierstrass theorem implies that (randomized) continuous stopping rules can be approximated by (randomized) signature stopping rules, given that

$$\mathbb{E}[Y_\tau] \leq \mathbb{E}\left[\|Y\|_\infty \right] < \infty.$$

Let, for simplicity, $Z \sim \mathrm{Exp}(1)$. Then we end up with

$$\sup_{\tau \in \mathcal{S}} \mathbb{E}\left[Y_{\tau \wedge T}\right] = Y_0 + \sup_{\ell \in \mathcal{W}_{d+1}} \mathbb{E}\left[\int_0^T \exp\left(-\int_0^t \left\langle \ell, \widehat{\mathbb{X}}^{<\infty}_{0,s} \right\rangle^2 \mathrm{d}s\right) \mathrm{d}Y_t\right].$$

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- Recalling that $\widehat{X}_s = (s, X_s)$, we have

$$\int_0^t \left\langle \ell, \widehat{\mathbb{X}}^{<\infty}_{0,s} \right\rangle^2 \mathrm{d}s = \int_0^t \left\langle \ell \sqcup\!\sqcup\, \ell, \widehat{\mathbb{X}}^{<\infty}_{0,s} \right\rangle \mathrm{d}s = \left\langle (\ell \sqcup\!\sqcup\, \ell)\mathbf{1}, \widehat{\mathbb{X}}^{<\infty}_{0,t} \right\rangle$$

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- Approximate exp by polynomials, giving the exponential shuffle $\exp^{\sqcup\!\sqcup}(\ell) := \sum_{n=0}^{\infty} \frac{1}{n!} \ell^{\sqcup\!\sqcup n}$.
- Often, Y can also be approximated by a linear functional on $\widehat{\mathbb{X}}^{<\infty}$. Otherwise, consider a RP extending $t \mapsto (t, X_t, Y_t)$. E.g., in the case $d = 1$, $Y \equiv X$, we obtain

$$\mathbb{E}\left[Y_{\tau_\ell \wedge T}\right] = \left\langle \exp^{\sqcup\!\sqcup}\left(-(\ell \sqcup\!\sqcup \ell)\mathbf{1}\right)\mathbf{2}, \mathbb{E}\left[\widehat{\mathbb{X}}^{<\infty}_{0,T}\right]\right\rangle \approx \left\langle \exp^{\sqcup\!\sqcup}\left(-(\ell \sqcup\!\sqcup \ell)\mathbf{1}\right)\mathbf{2}, \mathbb{E}\left[\widehat{\mathbb{X}}^{\leq N}_{0,T}\right]\right\rangle.$$

Theorem

Let $\mathbb{E}\left[\|Y\|_\infty\right] < \infty$. *Given* $\kappa > 0$, *define the stopping time* $\sigma = \sigma_\kappa$ *by* $\sigma := \inf\left\{ t \geq 0 \,\middle|\, \|\widehat{\mathbb{X}}\|_{p-\text{var};[0,t]} \geq \kappa \right\} \wedge T$. *Then,*

$$\sup_{\tau \in \mathcal{S}} \mathbb{E}\left[Y_{\tau \wedge T}\right] = \mathbb{E}\left[Y_0\right] + \lim_{\kappa \to \infty} \lim_{K \to \infty} \lim_{N \to \infty} \sup_{|\ell| + \deg(\ell) \leq K} \mathbb{E}\left[\int_0^{\sigma_\kappa} \left\langle \exp^{\sqcup\!\sqcup}(-(\ell \sqcup\!\sqcup \ell)\mathbf{1}), \widehat{\mathbb{X}}^{\leq N}_{0,t} \right\rangle \mathrm{d}Y_t\right].$$

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If Y *is a linear functional of* $\widehat{\mathbb{X}}^{<\infty}$, *this formula can be further simplified. E.g., if* $d = 1$ *and* $Y = X$, *then*

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1. Optimal stopping of Brownian motion X: By Fawcett's formula,

$$\mathbb{E}\left[\widehat{\mathbb{X}}_{0,T}^{<\infty}\right] = \exp\left(T\left(e_1 + \frac{1}{2}e_2 \otimes e_2\right)\right).$$

We immediately see that $\left\langle \exp^{\sqcup\!\sqcup}(-(\ell \sqcup\!\sqcup \ell)\mathbf{1})\mathbf{2}, \mathbb{E}\left[\widehat{\mathbb{X}}_{0,T}^{\leq N}\right]\right\rangle = 0.$

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2. Obtain approximately optimal strategy, not just approximation to value function. Let $\ell^* = \ell^*_{\kappa,K,N}$ an optimizer in the theorem. Construct

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- $\mathbb{E}\left[Y_{\tau^r_{\ell^*}}\right] \approx \mathbb{E}[Y_0] + \left\langle \exp^{\sqcup\!\sqcup}(-(\ell^* \sqcup\!\sqcup \ell^*)1)2,\ \mathbb{E}\left[\widehat{\mathbb{X}}^{\leq N}_{0,\sigma_\kappa}\right]\right\rangle \approx \sup_{\tau\in\mathcal{S}} \mathbb{E}\left[Y_{\tau\wedge T}\right]$
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3. Dual method based on minimization of martingales.

Recall that $\mathbb{L}^{<\infty}_{s,t} := \log \mathbb{X}^{<\infty}_{s,t} \in \mathfrak{g}(\mathbb{R}^d)$ and $\mathbb{L}^{\leq N}_{s,t} := \log \mathbb{X}^{\leq N}_{s,t} \in \mathfrak{g}^N(\mathbb{R}^d)$.

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- The log-signature $\mathbb{L}^{\leq N}_{s,t}$ contains the same information as $\mathbb{X}^{\leq N}_{s,t}$, but removes algebraic redundancies.
- No shuffle identity holds for (truncated) log-signatures, but $\dim \mathfrak{g}^N(\mathbb{R}^d) \ll \dim T^N(\mathbb{R}^d)$. E.g., for $d = 3$, $N = 6$: 196 vs 1092.

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- Use of the shuffle identity is not free, but often translated into very high degrees of truncation. E.g., suppose that $\deg = 3$ contains enough information, but a polynomial of degree 3 is to be linearized. Hence, the truncation degree $N = 9$ is required. (For $d = 3$, this leads to a dimension $\dim T^9(\mathbb{R}^3) = 29524$ – compare with $\dim T^3(\mathbb{R}^3) = 39$, $\dim \mathfrak{g}^3(\mathbb{R}^3) = 14$.)

Signatures are useful as features when their algebraic properties are efficiently used. Otherwise, log-signatures are probably preferable.

A class of fully connected Artificial Neural Networks

Given $K, q, I \in \mathbb{N}$ and an activation function φ (i.e., continuous, non-polynomial), let $\mathrm{DNN}(K, q, I; \varphi)$ denote the set of fully connected artificial neural networks with I hidden layers of dimension q, input dimension K and output dimension 1, i.e., for $\vartheta \in \mathrm{DNN}(K, q, I; \varphi)$ there are affine maps $A_0 : \mathbb{R}^K \to \mathbb{R}^q$, $A_1, \dots, A_{I-1} : \mathbb{R}^q \to \mathbb{R}^q$, $A_I : \mathbb{R}^q \to \mathbb{R}$ s.t.,

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Deep signature stopping rule

Given $\vartheta \in \mathrm{DNN}(K, q, I; \varphi)$ with $K = \dim \mathfrak{g}^N(\mathbb{R}^d)$ for some N, we define a deep signature stopping rule by

$$\tau_\vartheta := \inf\left\{ t \in [0, T] \,\middle|\, \int_0^t \vartheta\left(\mathbb{L}^{\leq N}_{0,s}\right)^2 \mathrm{d}s \geq 1 \right\}.$$

Let $\mathcal{T}_{\text{log}} := \bigcup_{N,q,I \in \mathbb{N}} \mathrm{DNN}\big(\dim \mathfrak{g}^N(\mathbb{R}^d), q, I; \varphi\big)$.

Theorem

If $\mathbb{E}\left[||Y||_\infty\right] < \infty$, we have

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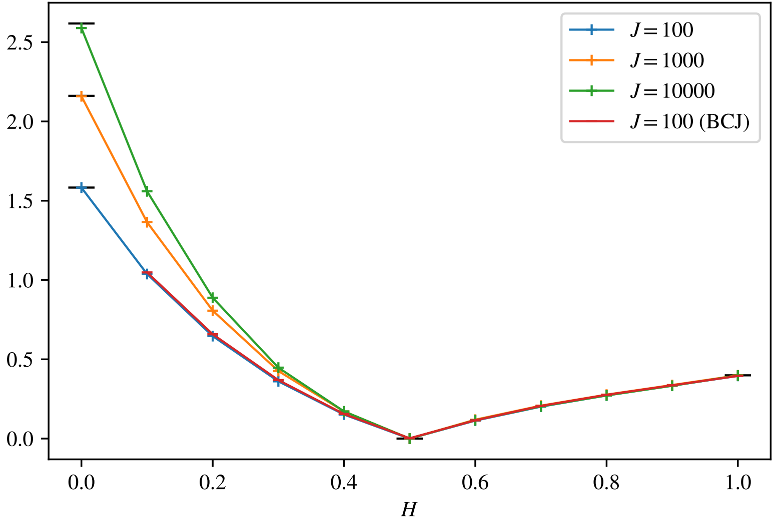
If $\mathbb{E}\left[\|Y\|_\infty\right] < \infty$, we have

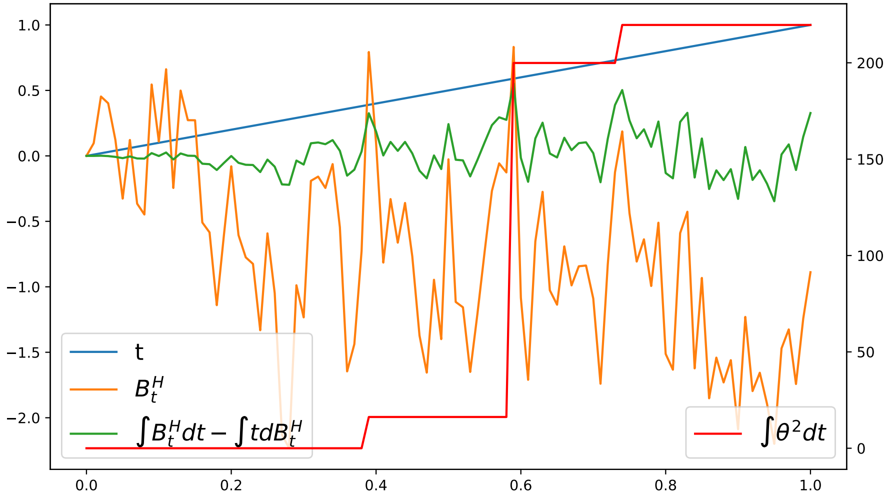
$$\sup_{\tau \in \mathcal{S}} \mathbb{E}\left[Y_{\tau \wedge T}\right] = \sup_{\vartheta \in \mathcal{T}_{\text{log}}} \mathbb{E}\left[Y_{\tau^r_\vartheta \wedge T}\right].$$

- Proof: Combination of the classical universal approximations theorem for neural networks and our earlier arguments.

Example: Optimal stopping of fractional Brownian motion $(W_t^H)_{t\in[0,1]}$ – approximate values



Example: Optimal stopping of fractional Brownian motion $(W_t^H)_{t\in[0,1]}$ – sample strategy



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